METRIC FRAÏSSÉ THEORY

Goals:

- Introduce Fraïssé constructions (& applications)
- Give a general & modern presentation (metric structures)
- Introduce logical concepts.

Introduction

Cantor (1895): $(\mathbb{Q}, <)$ is the only ctble linear dense order with no endpoints.

no max
min

$\exists$

Lead to key notions in logic

ω-categoricity

Back-and-Forth (Ehrenfeucht-Fraïssé games)

Fraïssé theory (50s)

<u>Universality</u>

any finite total order embeds in $\mathbb{Q}$

<u>Homogeneity</u>

<u>Def</u>: A structure M is homogeneous if any partial isomorphism

$$A \subseteq M \xrightarrow{\sim} B \subseteq M \quad A, B \text{ f.g.}$$

can be extended to an automorphism $M \xrightarrow{\sim} M$

$$\cong$$

$\mathrm{Age}(M) = \{ A \text{ f.g. structures} \mid \exists i : A \hookrightarrow M \text{ embedding} \}.$

e.g. $\mathrm{Age}(\mathbb{Q}, <) = \{\text{finite total orders}\}.$
$= \mathrm{Age}(\mathbb{Q} \sqcup \{*\})$

$\mathrm{Age}(\mathbb{F}^{\aleph_0}) = \{\text{finitely dimensional } \mathbb{F}\text{-vector spaces}\}.$

Fraïssé correspondence:

Countable $\xrightarrow{\mathrm{Age}}$ K class of f.g. structures hom. structure $\leftarrow$ Fraïssé limit

- hereditary

  $A \in K$

  $B \hookrightarrow A$

  $B \in K$

- closed under $\cong$
- essentially ctble
- directed

$$\begin{tikzcd} & \exists B \in K & \\ A_1 \in K \arrow[ur, dotted, hook] & & A_2 \in K \arrow[ul, dotted, hook'] \end{tikzcd}$$

- amalgamation ppty

$$\begin{tikzcd} & C \in K & \\ A_1 \in K \arrow[ur, dotted, hook] & & A_2 \in K \arrow[ul, dotted, hook'] \\ & B \in K \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

$$\begin{tikzcd} M \arrow[rr, "\exists \text{iso}"] & & M \\ A_2 \arrow[u, dotted, dash] \arrow[urr, dash] & & A_2 \arrow[u, dotted, dash] \arrow[ull, dash] \\ & B \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

I - The setting

Def: A signature $\mathcal{L}$ is a ctble set of symbols of two types: d of arity 2.

- relation symbols
- function symbols

$\exists$ come with an arity (a natural numbers)

<u>Def</u>: A (metric) <u>($\mathcal{L}$-)structure $\mathcal{M}$</u> is the data of:

- A metric space (M, d^M) which is complete
- For each relation symbol $R \in \mathcal{L}$ of arity n, a continuous map
$$R^M : M^n \longrightarrow \mathbb{R}$$
- For each function symbol $F \in \mathcal{L}$ of arity n, a continuous map
$$F^M : M^n \longrightarrow M$$

<u>Example</u>: Banach spaces

$$\mathcal{L} = \{d, \|\cdot\|, +, (\lambda \cdot \cdot)_{\lambda \in \mathbb{Q}}\}.$$

E - Banach space

$$d^M : M^{2} \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto \|x-y\|$$

$$\|\cdot\|^M : M^{1} \longrightarrow \mathbb{R}$$
$$x \longmapsto \|x\|$$

$$+^M : M \times M \longrightarrow M \quad \text{arity } 2.$$
$$(x,y) \longmapsto x+y$$

$$\lambda \cdot \_ : M \longrightarrow M$$
$$x \longmapsto \lambda x$$

$$\boxed{\text{We fix } \mathcal{L}}$$

Def: A, M two structures

$i : A \longrightarrow M$ is an <u>embedding</u> if

- $\forall R \in \mathcal{L}$ - relation symbol of arity n
  $\forall \bar{a} \in A^n$, $R^M(i(\bar{a})) = R^A(\bar{a})$
  (in particular, i is an isometry)
- $\forall F \in \mathcal{L}$ - function symbol of arity n,

$$\forall \bar{a} \in A^n,\ F^M(i(\bar{a})) = i(F^A(\bar{a}))$$

An <u>Isomorphism</u> is a surjective embedding.

$$\mathrm{Age}(M) = \{A \text{ finitely generated structures s.t. } \exists\, i : A \hookrightarrow M \text{ embedding}\}$$

A is f.g. if $\exists\, \bar{a} \in A^n$ s.t.

$$A = \langle \bar{a} \rangle$$

$\downarrow$

smallest substructure of A containing $\bar{a}$

<u>eg</u>: - if $\nexists$ function symbol in $\mathcal{L}$,

f.g. = finite

. For Banach spaces f.g = fin. dim.

Def: M-structure is **approximately homogeneous** if $\forall \bar{a}, \bar{b} \in M$ tuples and $\sigma : \langle \bar{a} \rangle \xrightarrow{\sim} \langle \bar{b} \rangle$, $\forall \varepsilon > 0$

$$\bar{a} \longmapsto \bar{b}$$

$\exists \tilde{\sigma} \in \mathrm{Aut}(M)$, $\quad d^M(\tilde{\sigma}(\bar{a}), \bar{b}) < \varepsilon$

$$\left(d^M(\bar{a}, \bar{b}) = \max_{i=1}^{n} d(a_i, b_i) \right)$$

eg: • $L^2(0,1)$ is exactly homogeneous

• $L^p(0,1)$ for $p \neq 4, 6, 8, \ldots$ is app. homogeneous. (Lusky)

• Urysohn space

• Gurarij space

• $L_p(0,1)$ as Banach lattices are homogeneous

app

Given M separable homogeneous structure, $\mathrm{Age}(M)$ satisfies:

- heredity
- closed under iso.
- directedness
- near amalgamation ppty (NAP)

$$\begin{tikzcd}
 & A_1 \arrow[dr, dashed, "j_1"] & \\
\langle \bar{a} \rangle \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & \circlearrowright \varepsilon & C \\
 & A_2 \arrow[ur, dashed, "j_2"'] &
\end{tikzcd}$$

$$d(j_2 i_2 \bar{a}, j_1 i_1 \bar{a}) < \varepsilon$$

- separability, completeness, continuity?

II - Types & type spaces

Given $\bar{a} \in A$ $\bar{b} \in B$ two tuples of same length

$\bar{a} \simeq \bar{b}$ if $\begin{matrix} \bar{a} \mapsto \bar{b} \\ a_i \mapsto b_i \end{matrix}$ defines an isomorphism $\langle \bar{a} \rangle \to \langle \bar{b} \rangle$

we also write $tp(\bar{a}) = tp(\bar{b})$

Take K a class of f.g. structures

I a finite set of variables.

$$S_I(K) = \{ \bar{a} \in A^I \mid A \in K \} / \cong$$

If $\bar{a} \in A^I$ $\quad \bar{a} : I \to A \quad I = \{x_1, \dots, x_n\}$.

$$tp(\bar{a}) : \varphi(x_1, \dots x_n) \mapsto \varphi^A(a(x_1), \dots, a(x_n))$$

formula

written thanks to symbols in $\mathcal{L}$

$$tp(\bar{a}) : d(x_1, x_2) \mapsto d(a(x_1), a(x_2))$$

If $p = tp(\bar{a})$, write $d^p(x_1, x_2)$ for

$$d(a_1, a_2)$$

Define for $p, q \in S_{\underline{n}}(K)$

$$\partial(p,q) = \inf \left\{ d^A(\bar{a},\bar{b}) : \begin{array}{l} A \in K \\ p = tp(\bar{a}) \\ q = tp(\bar{b}) \end{array} \right\}.$$

2-types in $(\mathbb{Q},<)$

$\underline{x_1 < x_2}$ $\qquad$ $x_2 < x_1$ $\qquad$ $x_1 = x_2$

$\quad\ \| \qquad\qquad\qquad\quad \|$

$tp(\underline{0,1}) \qquad tp(1,0) \qquad a_1 \quad a_2 \ {}^{\|}\ tp(0,0)$

$b_2 \quad b_1$

__Fact:__ M - separable structure

$K = Age(M)$

For any $n \in \mathbb{N}$

$(S_n(K), \partial)$ is a separable complete metric space $\qquad$ (PP)

$\forall R$ - relation symbol

$S_n(K) \longrightarrow \mathbb{R}$ $\qquad$ is continuous

$tp(\bar{a}) \mapsto K \quad (a)$

$$S_{n+1}(K) \to \mathbb{R} \qquad (CP)$$

$$tp(\bar{a}, b) \mapsto d^{\langle \bar{a}, b \rangle}(F^{\langle \bar{a} \rangle}(\bar{a}), b)$$

is continuous $\forall F \in \mathcal{L}$ function symbol

Main Theorem (Ben Yaacov)

Let K be a class of f.g structures. TFAE

i) K is hereditary, directed, satisfies (NAP), and for all n, we have (PP) and (CP).

ii) There exists a unique separable homogeneous structure M s.t. $Age(M) = K$.

III - Proof of the Fraïssé correspondence (due to Tsankov)

X ctble set.

Define $S_X(K) = \varprojlim_{\substack{I \subseteq X \\ \text{finite}}} S_I(K)$.

$S_X(K)$ (it is a completely metrizable space, Polish)

$$X \subseteq Y \rightsquigarrow S_Y(K) \to S_X(K)$$
$$p \mapsto p|_X$$

The restriction maps are continuous

Lemma (Realisability)

Let $p \in S_X(K)$, $|X| \leq \aleph_0$

There exists M a separable structure, $a \in M^X$

with $Age(M) \subseteq K$ s.t. for all finite

$F \subseteq X$, $\quad tp(a|_F) = p|_F$

(we write $tp(a) = p$)

Def: X, Y, $|X|, |Y| \in \aleph_0$, $p \in S_X(K)$, $q \in S_Y(K)$. A joining of p and q is an element $r \in S_{X \sqcup Y}(K)$ s.t.

$$r|_X = p \text{ and } r|_Y = q.$$

$$J(p,q) = \{\text{joinings of } p \text{ and } q\} \subseteq S_{X \sqcup Y}(K)$$

Rk: $\partial(p,q) = \inf\{d^r(X,Y) : r \in J(p,q)\}$

$p \in S_X(K)$, $q \in S_Y(K)$ $\quad |X| = |Y|$

$X \cap Y = \emptyset$

<u>Crucial Lemma</u>: Let X, Y ctble sets

$I \subseteq X$, $J \subseteq Y$, finite subsets.

$|I| = |J|$, $\varepsilon > 0$.

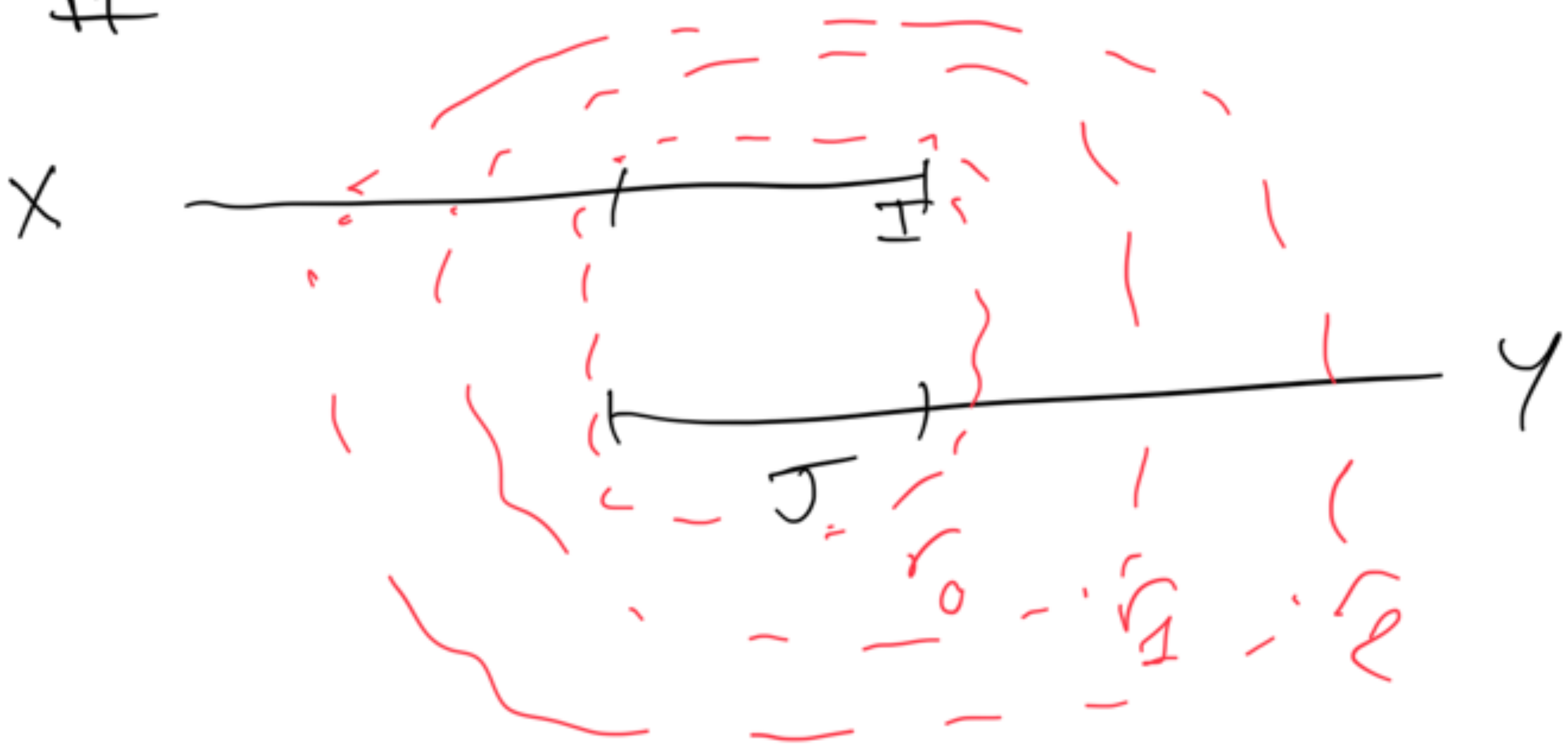
Let $p \in S_X(K)$ and $q \in S_Y(K)$

with $\partial(p|_I, q|_J) < \varepsilon$

Then there exists $r \in J(p,q)$ s.t.

$$d^r(I,J) < \varepsilon$$

<u>pf</u>:

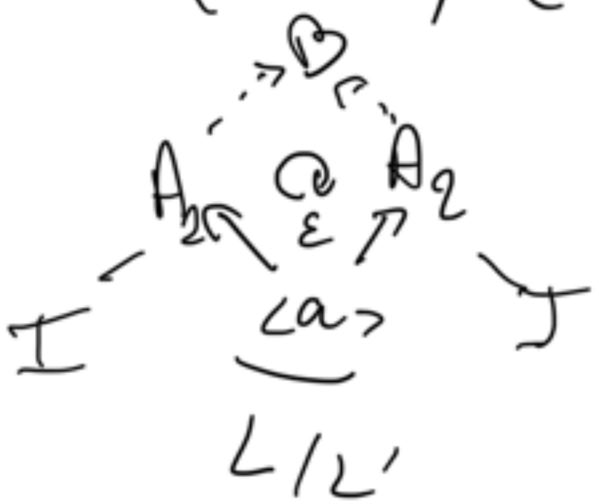


$$I = I_0 \subseteq I_1 \subseteq \dots \subseteq X$$

$J = J_0 \subseteq J_1 \subseteq \dots \subseteq Y$

We build $r_n \in \mathcal{J}(p|_{I_n}, q|_{J_n})$ s.t.

- $d^{r_n}(I,J) < \varepsilon$
- $\partial(r_{n+1}|_{I_n \cup J_n}, r_n) < 2^{-n}$



(NAP) $\iff$ $\forall I, J$ finite $L \subseteq I$, $L' \subseteq J$, $|L| = |L'|$

$p \in S_I(K)$, $q \in S_J(K)$ if

"$p|_L = q|_{L'}$" then

$\exists r \in \mathcal{J}(p,q)$ with

$d^r(L, L') < \varepsilon$

<u>$n=0$</u> By Rk on ∂, there is

$r_0 \in \mathcal{J}(p|_I, q|_J)$ s.t. $d^{r_0}(I,J) < \varepsilon$

induction step:

Amalgamate $p|_{I_{n+1}}$ and r_n

We get $s \in J(p|_{I_{n+1}}, r_n)$

$$\begin{cases} d^s(I_n, I'_n) < 2^{-(n+1)} \\ d^s(I_n, I'_n) < \varepsilon - d^{r_n}(I,J) \end{cases}$$

Define $r'_n = s|_{I_{n+1} \cup J_n}$

- $\partial(r'_n|_{I_n \cup J_n}, r_n) \le d^s(I_n, I'_n) < 2^{-(n+1)}$
- $d^{r'_n}(I,J) \le d^s(I, I') + d^{r_n}(I,J)$
  $< \varepsilon$

We do a similar amalgamation between r'_n and $q|_{J_{n+1}}$

$\longrightarrow$ we get the r_n we wished for.

$r_n \in S_{I_n \cup J_n}(K)$

$\rightsquigarrow \underline{\hat{r}_n \in S_{X \cup Y}(K)}$

$\hat{r}_n$ is Cauchy. $\hat{r}_n \to r \in S_{X \cup Y}(K)$.

Moreover, $\forall n$, $r|_{I_n \cup J_n} \in \underline{J(p|_{I_n}, q|_{J_n})}$.

$\rightsquigarrow r \in J(p,q)$

and $d^r(I,J) \leq \varepsilon$ by continuity of $d : S_{2n}(K) \to \mathbb{R}$.

<u>Detour</u>: M is "K-existentially closed" if $\forall I, J$ finite, $\forall \bar{a} \in M^I$, $\forall q \in S_{I \cup J}(K)$ with $\partial(\mathrm{tp}(\bar{a}), q|_I) < \varepsilon$,

there exists $\bar{b} \in M^J$ s.t.

$$\partial(\mathrm{tp}(\bar{a}, \bar{b}), q) < \varepsilon.$$

Def: $p \in S_X(K)$ is K-ec. if $\forall$ finite $I \subseteq X$, J, $J \cap X = \emptyset$ and $\forall q \in S_{I \cup J}(K)$ with $\partial(p|_I, q|_I) < \varepsilon$. there exists $J' \subseteq X$ with $|J'| = |J|$ s.t.

$$\partial(p|_{I \cup J'}, q) < \varepsilon.$$

It is stronger than K-ec of M_p as we require to to be contained in the infinite tuple of type p.

Proposition (*): $|X| = \aleph_0$, $p \in S_X(K)$.

TFAE:

i) p is K-ec

ii) $Age(M_p) = K$, M_p is homogeneous, and p enumerates M_p i.e. if $\bar{a}$ realises

p, then $\{a_0, a_1, \dots\}$ is dense in $\langle \bar{a} \rangle$

Note: If $\exists p \in S_X(K)$ e.c., then the main theorem follows.

Lemma: $p \in S_X(K)$, p e.c., $|Y| \leq \aleph_0$, $Y \cap X = \emptyset$, $q \in S_Y(K)$.
For comeagerly many $r \in J(p,q)$, there is an embedding

$$\langle a|_Y \rangle \subseteq \langle a|_X \rangle$$ where a is any tuple realising r.

Sketch of the proof:

It suffices to show that

$$\bigcap_{i \in Y} \bigcap_{n \in \mathbb{N}} \{ r \in J(p,q) : \exists i' \in X, \ d^r(i,i') < \varepsilon^{-n} \}.$$

is comeager $\qquad V_{i,m}$ dense ?

Basic open sets: $\mathcal{U} = \{ r \in J(p,q) : \partial(r|_{I\cup J}, r_0|_{I\cup J}) < \delta \}$.

$I \subseteq Y$ finite, $J \subseteq X$, $\delta > 0$. $r_0 \in J(p,q)$.

We may assume $\delta < 2^{-n}$, $i \in I$.

p K-ec $\Rightarrow$ $\exists I'$, $|I'| = |I|$,

$$\partial(p|_{J\cup I'}, r_0|_{J\cup I}) < \delta$$

Now, let us take $r_1 \in J(p, r_0)$

$$\subseteq S_{\underbrace{X_1}_{p} \cup \underbrace{X_2 \cup Y_2}_{r_0}}$$

$$d^{r_1}(\underbrace{J_1 \cup I_1'}_{\subseteq X_1}, \underbrace{J_2 \cup I_2}_{X_2 \cup Y_2}) < \delta$$

$r = r_1|_{X_1 \cup Y_1} \in J(p,q)$

$$\cdot\ \partial(r|_{I\cup J}, r_0|_{I\cup J}) \le d^{r_2}(I_2\cup J_2, I_2\cup J_2) \le d^{r_1}(I_1, J_1) < \delta$$

For all $i'\in I'$, we have

$$\cdot\ d^r(i,i') \le d^r(I, I') = d^{r_2}(I_2, I_1') < \delta < 2^{-n}$$

$\square$

Corollary: $|X| = \aleph_0$

1. $p\in S_X(K)$ K-ec, A separable with $Age(A)\subseteq K \Rightarrow A \subseteq M_p$
2. $p_1, p_2 \in S_X(K)$, p_1, p_2 K-ec. $\Rightarrow M_{p_1} \simeq M_{p_2}$.

proof of Prop (*), (i) $\Rightarrow$ (ii)

Let $p \in S_X(K)$, K-ec, $|X| = \aleph_0$.

- $\forall A \in K$, $\mathrm{Age}(A) \subseteq K \rightsquigarrow A \subseteq M_p$.

  We deduce that $\mathrm{Age}(M_p) \supseteq K$.

- The fact that p enumerates M_p comes from continuity of function symbols.

- <u>Homogeneity</u>: $I, J \subseteq X$, $\partial(p|_I, p|_J) < \varepsilon$.

$\mathcal{U} = \{ r \in \mathcal{J}(p,p) : d^r(I_1, J_2) < \varepsilon \}$. open.

$\subseteq S_{X_1 \cup X_2}(K)$

Thus, there exists $r \in \mathcal{U}$ s.t. if

$a \in M^*$, $b \in M^*$, $r = \mathrm{tp}(a\, b)$, we have

an isomorphism $f : M_p \simeq \langle a \rangle \longrightarrow \langle b \rangle \simeq M_p$

st. $d^{M_p}(f(I), J) = d^r(I_1, J_2) < \varepsilon$.

$\square$

<u>Proposition</u>: $|X| = \aleph_0$ $\{p \in S_X(K) : p \text{ is } K\text{-ec}\}$

is dense G_δ in $S_X(K)$

is dense G_δ in $S_X(K)$.

Hint: $\{K\text{-ec}\} = \bigcap_{\substack{I \subseteq X \\ \text{finite}}} \bigcap_{\substack{J \text{ finite} \\ \subseteq \mathbb{N}}} \bigcap_{\varepsilon > 0} \bigcap_{q \in D_{J \cup I}} \{p : \partial(p|_I, q|_I) \geq \varepsilon \text{ or } \exists J' \subseteq X \ \partial(p|_{I \cup J'}, q) < \varepsilon\}$.

$D_J \subseteq S_J(K)$ dense countable set

$\swarrow$

G_δ dense

Applications

$\mathrm{Aut}(M) = G \curvearrowright M = \mathrm{Flim}(K)$

$\uparrow$

$A \in K = \mathrm{Age}(M)$

Examples:

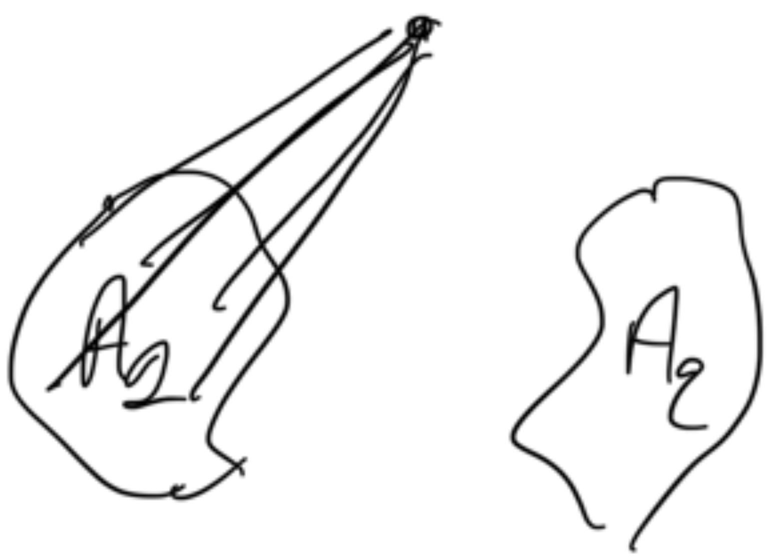
- $K = \{\text{finite linear orders}\}$

  $M = (\mathbb{Q}, <)$

- $K = \{\text{finite graphs}\}$

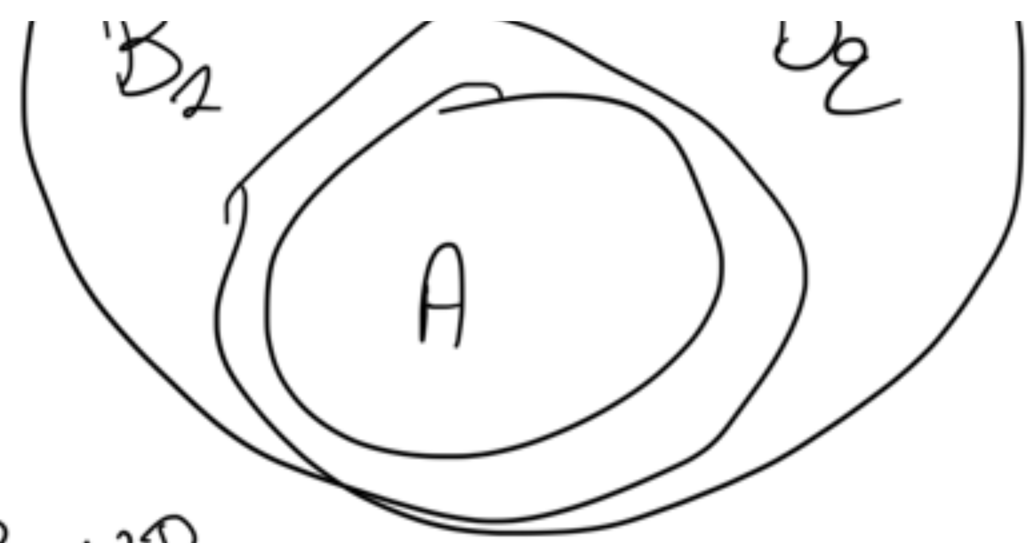
  $M =$ Rado graph

$\mathbb{N}$



- $K = \{\text{finite metric spaces}\}$

  $M =$ the Urysohn space



$C = B_1 \cup B_2$

$$d^C(x,y) = \begin{cases} \text{the usual distance} & \text{if } x,y \in B_i \\ \inf_{a \in A} d^{B_1}(x,a) + d^{B_2}(y,a) & \text{if } x \in B_1 \setminus A,\ y \in B_2 \setminus A \end{cases}$$

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rational Urysohn space

$K = \{ \text{finite metric spaces} \mid d \text{ is } \mathbb{Q}\text{-valued} \}$

$\sim$ encoding

- $K = \{ \text{finite dim. Hilbert spaces} \}$

$H_2 \cap H_3$
H_2

$$\underbrace{H_1^{\perp 2} \oplus H_2 \oplus H_1^{\perp 3}}$$

$M = L^2(0,1)$

- $\mathcal{K} = \{$ finite dimensional Banach spaces $\}$

$M =$ Gurarij Space

- $L^p(0,1)$ is homogeneous for all $p \neq 4, 6, 8, \dots$

as a Banach space

- $\mathcal{K} = \{$ finite probability measure algebra $\}$

$M =$ atomless separable proba measure algebra

$= \mathrm{MAlg}([0,1])$

Motivations:

(1) Studying $\mathcal{K}$ via M or vice-versa

↳ genericity

↳ combinatorics (KPT)

(2) Studying a Polish group G by viewing it as $\mathrm{Aut}(M)$.

Thm (Melleray) Any Polish group is isomorphic to $\mathrm{Aut}(M)$ for some separable homogeneous metric structure M on a ctble signature $\mathcal{L}$.

Rk: If G is non-Archimedean, then M can be chosen discrete.

$$G \underset{\text{closed}}{\leq} S_\infty \qquad G \curvearrowright \mathbb{N}^n$$

$$\mathcal{L} = \{ R_{n,\mathcal{O}} : n \in \mathbb{N},\ \mathcal{O} \text{ orbit of } G \curvearrowright \mathbb{N}^n \}.$$

M_G $\mathcal{L}$-structure $\quad M_G = \mathbb{N}$

$$R_{n,G}^{M_G}(\bar{a}) = \begin{cases} 1 & \text{if } \bar{a} \in G \\ 0 & \text{else} \end{cases}$$

n-tuple

$\mathrm{Aut}(M_G) = G \qquad G \leq \mathrm{Aut}(M_G) \leq S_\infty$

$\sigma \in \mathrm{Aut}(M_G)$. $\quad a_1, \dots, a_n \in M_G^n$

$\bar{a} \in G_{\bar{a}}$

$$\underbrace{R_{n,G_{\bar{a}}}^{M_G}(\sigma(\bar{a}))}_{\sigma(\bar{a}) \in G_{\bar{a}}} = R_{n,G_{\bar{a}}}^{M_G}(\bar{a}) = 1$$

$\exists g \in G$ s.t. $\sigma(\bar{a}) = g\bar{a}$

G is dense in $\mathrm{Aut}(M_G)$

$G = \overline{G} = \mathrm{Aut}(M_G)$.

of of the general fact:

Pf of the general fact:

$G \curvearrowright \hat{G}_L$ – left-completion of G with respect to an left-invariant metric on G.

$$\mathcal{L} = \{R_{n,G}\}_{n\in\mathbb{N}}$$
$$G \in \{\overline{G\cdot\bar{a}}\}_{\bar{a}\in\hat{G}_L^n}$$

$$M_G = \hat{G}_L \qquad R_{n,G}^{M_G}(\bar{a}) = d_L(\overline{G\cdot\bar{a}}, G)$$

I – KPT Correspondence

G – Polish group.

Def: G is extremely amenable if for any continuous action $G \curvearrowright K$ on a compact Hausdorff space, $\exists x \in K$ s.t. $G\cdot x = \{x\}$.

G is amenable

$\exists\mu$ Borel

proba measure on K s.t.

$$\forall f \in C(X), \int_X f(g^{-1}x)\, d\mu(x) = \int_X f\, d\mu$$

Q: How to relate these properties with properties of $M/\mathcal{K}$?

(Technical assumption: Lipschitz)

$A = \langle \bar{a} \rangle \in \mathcal{K}$ $B \in \mathcal{K}$ (or $B = M$).

$\mathrm{Emb}(\bar{a}, B) = \{\alpha : A \to B \text{ embeddings}\}$.
with metric $\rho_{\bar{a}}(\alpha, \beta) = d^B(\alpha(\bar{a}), \beta(\bar{a}))$

$\mathrm{Emb}(A, B) = \{\alpha : A \to B \text{ embeddings}\}$.

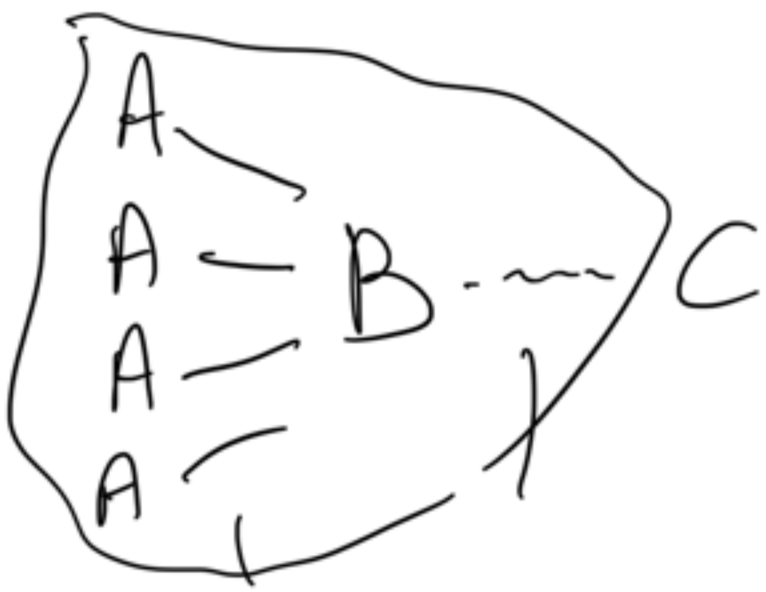
A coloring of $\mathrm{Emb}(\bar{a}, B)$ is a 1-Lipschitz map $c : \mathrm{Emb}(\bar{a}, B) \to [0, 1]$.

Def: $\mathcal{K}$ has the app. Ramsey Prop (ARP) (resp. metric convex RP) if for all $\langle \bar{a} \rangle = A, B \in \mathcal{K}$, all finite $F \subseteq \mathrm{Emb}(\bar{a}, B)$, all $\varepsilon > 0$, there exists $C \in \mathcal{K}$ s.t. $\forall c$ coloring of $\mathrm{Emb}(\bar{a}, C)$, there exists $\beta \in \mathrm{Emb}(B, C)$ (∀ finitely supp. proba measure on $\mathrm{Emb}(B, C)$) s.t.

$$|c(\beta \circ \alpha) - c(\beta \circ \alpha')| < \varepsilon$$

for all $\alpha, \alpha' \in F$.

$$\left| \int_{\mathrm{Emb}(B,C)} c(\beta \circ \alpha)\, d\nu(\beta) - \int c(\beta \circ \alpha')\, d\nu(\beta) \right| < \varepsilon$$



<u>Thm</u> (Ramsey) The class of finite

linear orders has the (approximate) Ramsey property.

<u>Thm</u> (Kechris–Pestov–Todorcevic, Melleray–Tsankov)

TFAE:

i) G is extremely amenable

ii) K has the (ARP)

G, $\bar{a} \in M^n$. we can define a pseudometric $d_{\bar{a}}$ on G

$$d_{\bar{a}}(g,h) = d^M(g\bar{a}, h\bar{a})$$

The family $\{d_{\bar{a}}\}$ defines the left uniform structure on G.

Fact: G is ext. amenable iff

$G \curvearrowright (G, d_{\bar{a}})$ is finitely osc. stable.

$\forall F \subseteq G$, $\forall \varepsilon > 0$, $\forall f : \underbrace{(G, d_{\bar{a}})}_{\text{almost } \mathrm{Emb}(\bar{a}, M)} \to [0,1]$

1-Lipschitz, $\exists g$ s.t.

$$|f(x) - f(y)| < \varepsilon$$

$\forall x, y \in gF$.

<u>Thm</u> (Moore, Kaichouri)

G is amenable iff $\mathcal{K}$ has the metric convex Ramsey property.

<u>Example</u>: Bartošová, López-Abad, Lupini and MBombo (21)

$\mathrm{Iso}(\mathbb{G})$ is extremely amenable.

II - Genericity

$\mathcal{C}$ = Cantor space $\quad G$ - countable group

$A_{min}(G) = \{$ minimal continuous actions $G \curvearrowright \mathcal{C} \}$.

$\hookrightarrow \forall \mathcal{U} \subseteq \mathcal{C}$ open $\exists g_1, \dots g_n \in G$ s.t. $\mathcal{C} = \bigcup g_i \mathcal{U}$.

$\mathrm{Homeo}(\mathcal{C}) \curvearrowright A_{min}(G)$ by conjugacy

$A_{minpmp}(G) = \{ G \curvearrowright \mathcal{C}$ minimal and pmp $\}$.

$d \geq 2$.

Thm (Doucha - Melleray - Tsankov)

The space $A_{min}(F_d)$ has a comeager conjugacy class

The space $A_{minpmp}(F_d)$ has a comeager conjugacy class.